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Note

A fixed-point theorem in a category of compact metric spaces ☆

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Abstract

Various results appear in the literature for deriving existence and uniqueness of fixed points for endofunctors on categories of complete metric spaces. All these results are proved for *contracting* functors which satisfy some further requirements, depending on the category in question.

Following a new kind of approach, based on the notion of η-*isometry*, we show that the sole hypothesis of contractivity is enough for proving existence and uniqueness of fixed points for endofunctors on the category of compact metric spaces and embedding-projection pairs.

1. Introduction

Categories of metric spaces have turned out to be very useful in giving denotational semantics to concurrent programming languages. The key idea is the following: the longer the processes exhibit the same behaviour, the smaller distance between two processes is. In various papers (see e.g. [2, 3, 12, 14, 17]) mathematical theories are developed for solving domain equations of the form $X = FX$, where F is a functor, in categories of complete metric spaces. These can be viewed as possible categorical versions of the Banach–Caccioppoli's fixed-point theorem in complete metric spaces. All the results apply to *contracting* functors, for which the equation $X = FX$ has a fixed point. In order to obtain *uniqueness*, further hypotheses have to be added. In [3, 12] three approaches are presented, all other results on uniqueness of fixed points in categories of complete metric spaces appearing in the literature (see e.g. [14, 17]) are

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based on them. The first approach [3] deals with endofunctors which are contracting and *hom-contracting* in the category of complete metric spaces and embedding-projection pairs. The second approach [3] deals with contracting endofunctors in a *base-point* category of complete metric spaces, i.e. a category in which each space has a specially designated base-point and morphisms preserve the base-points. The third approach [12] deals with contracting functors such that $F(\emptyset) \neq \emptyset$, where $\emptyset$ is the *empty metric space*, in the category of complete metric spaces.

It is worth mentioning that the problem of finding a (unique) solution to $X = FX$, where F is not necessarily a functor, has also been faced in non-categorical settings; in [7] fixed-point results are presented in the framework of *hyperuniverses*.

In this note we prove that if one works in a category of *compact* metric spaces, then contractivity of F (without any extra hypothesis) is enough to obtain existence and uniqueness of solutions of domain equations.

We proceed as follows. First we introduce the notion of *η-isometry* which can be viewed as an isometry "up to a factor η". Then, using contractivity, we prove that two solutions M and M' of the equation $X = FX$ are η-isometric for each η. Finally, using the compactness hypothesis, we prove that M and M' are isometric.

A peculiarity of our technique is that isometries are proved without invoking, as in the case of the other approches, commutativity of categorical diagrams.

2. Mathematical preliminaries

In this section we give basic definitions and properties of metric spaces. As in [3] we consider only metric spaces with bounded diameter, i.e. the distance between two points never exceeds 1.

A sequence $(x_n)_n$ in a metric space (M, d) is called a *Cauchy sequence* whenever we have

$$\forall \varepsilon > 0.\ \exists n_0 \in \mathbb{N}.\ \forall n, m \geqslant n_0.\ d(x_n, x_m) < \varepsilon.$$

A metric space (M, d) is called *complete* whenever each Cauchy sequence converges to an element of M. It is called *compact* if each sequence contains a converging subsequence.

Let (M_1, d_1), (M_2, d_2) be metric spaces and let $A \geqslant 0$. $M_1 \to^A M_2$ denotes the set of functions $f : M_1 \to M_2$ which satisfy the condition:

$$\forall x, y \in M_1.\ d_2(f(x), f(y)) \leqslant A \cdot d_1(x, y).$$

The functions in $M_1 \to^1 M_2$ are called *non-distance increasing* (NDI). The elements of $M_1 \to^A M_2$, for $0 \leqslant A < 1$, are called *contracting functions* or *contractions*. A function $f : M_1 \to M_2$ is an *isometric embedding* if

$$\forall x, y \in M_1.\ d_2(f(x), f(y)) = d_1(x, y).$$

If f is a bijection then it is an *isometry*.

Now we recall the classical result, of existence and uniqueness of fixed point for contracting functions.

Theorem 2.1 (Banach–Caccioppoli's fixed-point theorem). *Let (M, d) be a complete metric space and $f: M \to M$ a contracting function. Then there exists a unique fixed point $fix(f)$ for f in M:*

$$fix(f) = \lim_{n \to +\infty} f^n(x_0), \quad x_0 \in M.$$

3. Fixed points in the category $\mathbf{CMS}^E$

In this section, following [3], we see how it is possible to generalize the Banach–Caccioppoli's fixed-point theorem to a categorical setting. One introduces first the category CMS^E of complete metric spaces and *embedding-projection pairs* and defines the concepts of converging tower, contracting and hom-contracting functor. Then one shows that a contracting functor F gives rise to a converging tower and that the limit of this tower is a fixed point for the functor, which solves therefore the equation $X = FX$. Finally one proves that hom-contractivity ensures *uniqueness* of fixed point.

Let M_1, M_2 be complete metric spaces. An *ep-pair (embedding–projection pair)* from M_1 to M_2 is a pair of functions $\iota = \langle i, j \rangle$ such that $i: M_1 \to M_2$ is an isometric embedding, $j: M_2 \to M_1$ is an NDI function and $j \circ i = id_{M_1}$. We denote by CMS^E the category whose objects are non-empty complete metric spaces and morphisms ep-pairs. Composition of morphisms is defined in the obvious way.

Notice that if there is a morphism $\iota = \langle i, j \rangle : M_1 \to M_2$ then we can consider M_1 as an approximation of M_2 since M_1 can be isometrically embedded into M_2. The measure of this approximation is given by

$$\delta(\iota) = d_{M_2 \to M_2}(i \circ j, id_{M_2}) \quad (= \sup_{y \in M_2} d_{M_2}(i(j(y)), y)).$$

A *tower* in CMS^E is a sequence $(M_n, \iota_n)_n$ of objects and morphisms such that for all n we have $\iota_n : M_n \to M_{n+1}$. It is called a *converging tower* if

$$\forall \varepsilon > 0.\ \exists n_0 \in N.\ \forall m > n \geqslant n_0.\ \delta(\iota_{nm}) < \varepsilon, \text{ where } \iota_{nm} = \iota_{m-1} \circ \cdots \circ \iota_n.$$

A converging tower is intuitively a sequence of spaces such that when n increases M_n approximates better and better M_{n+k} (for each integer k).

The following result gives a criterion for checking the initiality of a cone.

Lemma 3.1 (Initiality lemma). *Let $(M_n, \iota_n)_n$ be a converging tower in CMS^E and let $(M, (\gamma_n)_n)$, with $\gamma_n = \langle \alpha_n, \beta_n \rangle$, be a cone for that tower. Then*

$$(M, (\gamma_n)_n) \text{ is an initial cone iff } \lim_{n \to \infty} \delta(\gamma_n) = 0.$$

We now outline the direct limit construction in CMS^E. First of all we fix some notations. Let $(M_n, \iota_n)_n$ be a converging tower in CMS^E, where $\iota_n = \langle i_n, j_n \rangle$. Define $i_{nk}: D_n \to D_k$ as follows: if $n < k$ then $i_{nk} = i_{k-1} \circ \cdots \circ i_n$; if $n > k$ then $i_{nk} = j_k \circ \cdots \circ j_{n-1}$; if $n = k$ then $i_{nn} = id_{D_n}$.

The *direct limit* of $(M_n, \iota_n)_n$ is a cone $(M, (\gamma_n)_n)$, such that

– $M = \{(x_n)_n \colon \forall n \in N.\ x_n \in M_n \text{ and } x_n = j_n(x_{n+1})\}$. $d: M \times M \to [0,1]$ is defined as follows: for all $(x_n)_n, (y_n)_n$

$$d((x_n)_n, (y_n)_n) = \sup_{n \in \mathbb{N}} d_{M_n}(x_n, y_n).$$

– Morphisms $\gamma_n = \langle \alpha_n, \beta_n \rangle : M_n \to M$ are defined as follows:

$$\alpha_n : M_n \to M \qquad \alpha_n(x) = (x_k)_k \quad \text{where } x_k = i_{nk}(x);$$

$$\beta_n : M \to M_n \qquad \beta_n((x_k)_k) = x_n.$$

It is possible to show that the direct limit M is a complete metric space and $(M, (\gamma_n)_n)$ is a cone for the tower $(M_n, \iota_n)_n$. Moreover using the initiality lemma one can prove that $(M, (\gamma_n)_n)$ is an initial cone for the tower.

3.1. Fixed-point theorems

In this subsection we present the technique shown in [3] for solving domain equations $X = FX$ in CMS^E.

We start with the notion of *contractivity* for functors. A functor $F: \mathrm{CMS}^E \to \mathrm{CMS}^E$ is called *contracting* if there exists ε, $0 \leqslant \varepsilon < 1$, such that for each morphism $\iota: M_1 \to M_2$ the following inequality holds:

$$\delta(F\iota) \leqslant \varepsilon \cdot \delta(\iota).$$

The importance of contractivity arises when one considers a converging tower $(M_n, \iota_n)_n$ with an initial cone $(M, (\gamma_n)_n)$. In such a case, whenever F is contracting (see e.g. [3, Lemma 3.13]), $(FM_n, F\iota_n)_n$ is a converging tower with $(FM, (F\gamma_n)_n)$ as an initial cone. Moreover, if one starts from an initial ep-pair $\iota_0: M_0 \to FM_0$, then the tower $(F^n M_0, F^n \iota_0)_n$ is converging.

These remarks are essential in showing the theorem of existence of fixed points for domain equations in CMS^E.

Theorem 3.2 (Existence of fixed point). *Let $F: \mathrm{CMS}^E \to \mathrm{CMS}^E$ be a contracting functor. Then F has a fixed point, that is, there exists a complete metric space M such that $M \cong FM$.*

Proof (sketch). Consider the one-point metric space $M_0 = \{x_0\}$ and let $\iota_0: M_0 \to FM_0$ be any morphism. Building the tower $(F^n M_0, F^n \iota_0)_n$. This is a converging tower, thus it has a direct limit $(M, (\gamma_n)_n)$ which is an initial cone for the tower.

Moreover F preserves tower and its initial cone. This is enough (see [3, Theorem 3.14]) to conclude that $FM \cong M$. $\square$

In order to extend to the categorical setting the Banach–Caccioppoli's fixed-point theorem, we now turn our attention to *uniqueness* of fixed points. As remarked in the introduction, three methodologies have been introduced in the literature in order to obtain uniqueness. Before presenting the common strategy shared by them, let us fix some notations.

Let M' be a fixed point of F (F contracting), say $\lambda : M' \to FM'$, for an isometry λ. Let Δ denote the tower $(F^n\{p_0\}, F^n\iota_0)_n$, where $\{p_0\}$ is the one-point space and ι_0 is any ep-pair from $\{p_0\}$ to $F\{p_0\}$. The crucial idea of all approaches presented in the literature is the following:

(*) *if M' can be turned into a cone $(M', (\gamma_n)_n)$ of Δ, where*

(**) $\gamma_{n+1} = \lambda^{-1} \circ F\gamma_n \quad (n \in \mathbb{N})$

then M' is isometric to the direct limit of Δ (therefore the fixed point is unique up to isometry).

The result follows essentially from the Initiality lemma, by noticing that contractivity of F enforces $\delta(\gamma_n) \to 0$.

As an example we see how this idea works in the case of contracting and *hom-contracting* functors, (for a detailed explanation see [3]). First of all we recall the notion of *hom-contractivity*.

Definition 3.3. A functor $F : \mathrm{CMS}^E \to \mathrm{CMS}^E$ is called *hom-contracting* if for each M_1, M_2 in CMS^E there exists ε, $0 \leqslant \varepsilon < 1$, such that

$$F_{|hom(M_1,M_2)} \in hom(M_1, M_2) \to^{\varepsilon} hom(FM_1, FM_2).$$

We now show how hom-contractivity implies uniqueness of fixed points. According to the previous notation, we want to prove that M' is essentially unique. As shown, it is enough to prove that M' satisfies (*), that is there exist morphisms $\tilde{\gamma}_n : F^n\{p_0\} \to M'$ such that $(M', (\tilde{\gamma}_n)_n)$ is a cone for Δ (i.e. $\tilde{\gamma}_n = \tilde{\gamma}_{n+1} \circ F^n\iota_0$) and moreover (**) holds. An easy induction on n shows that (**) is equivalent to finding $\tilde{\gamma}_0 : \{p_0\} \to M'$ such that

$$\tilde{\gamma}_0 = \lambda^{-1} \circ F\tilde{\gamma}_0 \circ \iota_0.$$

Now $\tilde{\gamma}_0$ can be seen as the fixed point of the functional $\Phi : hom(M_0, M') \to hom(M_0, M')$ defined by

$$\Phi(u) = \lambda^{-1} \circ F(u) \circ \iota_0.$$

Since F is *hom-contracting* Φ has a fixed point, thus one can conclude the existence of $\tilde{\gamma}_0$. Therefore (*) holds and uniqueness is proved.

This discussion justifies the theorem of uniqueness of fixed point for contracting and *hom-contracting* functors.

Theorem 3.4 (Existence and uniqueness of fixed point). *Let $F:\mathrm{CMS}^E \to \mathrm{CMS}^E$ be a contracting and hom-contracting functor. Then F has a unique fixed point up to isomorphism, that is there exists a complete metric space M such that*
– $M \cong FM$;
– $\forall M'$ *in* CMS^E $FM' \cong M' \Rightarrow M \cong M'$.

4. The result

In this section we consider the full subcategory KMS^E of CMS^E whose objects are compact metric spaces. Our aim is to prove that in KMS^E equations $X = FX$ have a unique fixed point, provided that F is just contracting. It is interesting to point out that *our technique does not rely on satisfying* (*).

Some preliminary remarks about compact metric spaces are in order. First of all we recall that given two compact metric spaces M_1 and M_2, the space of NDI functions from M_1 to M_2, endowed with the metric

$$d(f,g) = \sup_{x\in M_1} d_2(f(x),g(x)),$$

is a compact metric space . This fact follows from the Ascoli-Arzelà's theorem (see [9, Theorem 7.17]). Moreover KMS^E is closed with respect to direct limit constructions. This is a consequence of Tychonoff's theorem on compactness of product spaces (see e.g. [9]). A direct proof is given in [17].

Using these properties and taking into account that every compact space is complete, it is possible to show that the results of the previous section follow also in the subcategory KMS^E, that is every [*hom-*]*contracting* functor $F:\mathrm{KMS}^E \to \mathrm{KMS}^E$ has a [unique] fixed point in KMS^E.

Now we prove that uniqueness follows from the sole hypothesis that F is contracting.

Let $F:\mathrm{KMS}^E \to \mathrm{KMS}^E$ be a contracting functor, and let $(M,(\gamma_n)_n)$ be the direct limit of the tower $(F^n M_0, F^n \iota_0)_n$, where $\gamma_n = \langle \alpha_n, \beta_n \rangle$. Let κ be the canonical isomorphism between M and FM. Let M' be another fixed point and λ an isomorphism between M' and FM'. Choose a morphism $\tilde{\gamma}_0 : M_0 \to M'$ and define for all n, $\tilde{\gamma}_{n+1} = \lambda^{-1} \circ F\tilde{\gamma}_n$. We know that $(M',(\tilde{\gamma}_n)_n)$ is not in general a cone for the tower $(F^n M_0, F^n \iota_0)_n$, but contractivity of F assures that $\delta(\tilde{\gamma}_n) \to 0$. Thus $F^n M_0$ approximates M' better and better when n increases and the same thing happens for M, since it is the limit of the tower.

In the compact case this is sufficient to conclude $M \cong M'$. Before going into technical details we explain briefly this point. Consider again $\gamma_n : F^n M_0 \to M$ and $\tilde{\gamma}_n = \langle \tilde{\alpha}_n, \tilde{\beta}_n \rangle$:

$F^n M_0 \to M'$. We define, for each n,

$$h_n = \tilde{\alpha}_n \circ \beta_n : M \to M',$$

$$k_n = \alpha_n \circ \tilde{\beta}_n : M' \to M.$$

The first remark above (stating that the space of NDI functions between *compact* metric spaces is compact) assures that we can find $h: M \to M'$ and $k: M' \to M$, limits of suitable subsequences of $(h_n)_n$ and $(k_n)_n$ respectively. Although $\langle h_n, k_n \rangle$ are not in general ep-pairs (for this reason we will be forced to introduce η-isometries), nevertheless $k = h^{-1}$. Therefore we obtain $M \cong M'$.

We now give the technical details. First of all we introduce the notion of *η-isometry* for $\eta \geqslant 0$, which generalizes that of ep-pair.

Definition 4.1. Let M_1 and M_2 be metric spaces and let $\eta \geqslant 0$. We say that M_1 and M_2 are *η-isometric* if there exists a pair of NDI functions $\gamma = \langle \alpha, \beta \rangle$, $\alpha: M_1 \to M_2$ and $\beta: M_2 \to M_1$, such that

$$d(\beta \circ \alpha, id_{M_1}) \leqslant \eta \quad \text{and} \quad d(\alpha \circ \beta, id_{M_2}) \leqslant \eta.$$

γ is called a *η-isometry*.

Notice that a 0-isometry is an isometry. One can wonder whether two spaces η-isometric for all $\eta > 0$ are isometric. This holds if the spaces are compact, but it is not true in general for complete metric spaces (see [1] for a counterexample). This fact is essential in our proof for deriving uniqueness of fixed points for contractive endofunctors over KMS^E. The question whether contractivity of functors forces uniqueness of fixed point in CMS^E is open.

Lemma 4.2. *Let M_1 and M_2 be compact metric spaces. If M_1 and M_2 are η-isometric for all $\eta > 0$ then M_1 and M_2 are isometric.*

Proof. For every $n \in N$, let $\gamma_n = \langle \alpha_n, \beta_n \rangle$ be a η_n-isometry between M_1 and M_2, with $\eta_n \to 0$. Now $(\alpha_n)_n$ is a sequence of NDI functions between compact spaces, hence (as remarked at the beginning of the section), there exists a subsequence $(\alpha_{n_k})_k$ converging to an NDI function $\alpha: M_1 \to M_2$. In the same way $(\beta_{n_k})_k$ contains a converging subsequence $(\beta_{n_{k_h}})_h$ with limit $\beta: M_2 \to M_1$.

We show that α is an isometry and $\beta = \alpha^{-1}$. To keep notation simple we define

$$(\alpha_{n_{k_h}})_h = (\alpha'_h)_h \quad \text{and} \quad (\beta_{n_{k_h}})_h = (\beta'_h)_h.$$

We have

$$\begin{aligned} &d(\beta \circ \alpha, id_{M_1}) \\ &\quad = d(\lim_{h_1 \to \infty} \beta'_{h_1} \circ \lim_{h_2 \to \infty} \alpha'_{h_2}, id_{M_1}) \\ &\quad = \lim_{h_1 \to \infty} \lim_{h_2 \to \infty} d(\beta'_{h_1} \circ \alpha'_{h_2}, id_{M_1}). \end{aligned}$$

Now notice that

$$
\begin{aligned}
& d(\beta'_{h_1} \circ \alpha'_{h_2}, id_{M_1}) \\
& \quad \leqslant d(\beta'_{h_1} \circ \alpha'_{h_2}, \beta'_{h_1} \circ \alpha'_{h_1}) + d(\beta'_{h_1} \circ \alpha'_{h_1}, id_{M_1}) \\
& \quad \leqslant d(\alpha'_{h_2}, \alpha'_{h_1}) + d(\beta'_{h_1} \circ \alpha'_{h_1}, id_{M_1}) \quad [\text{because } \beta'_{h_1} \text{ is NDI}].
\end{aligned}
$$

Since $(\alpha'_h)_h$ is a converging sequence (hence a Cauchy sequence) and every α'_h is a $\eta_{n_{k_h}}$-isometry, we can conclude that $d(\beta \circ \alpha, id_{M_1}) = 0$, hence

$$\beta \circ \alpha = id_{M_1}.$$

Similarly we establish that $\alpha \circ \beta = id_{M_2}$. Since α is an NDI bijection which has an inverse NDI function β, we can conclude that α is an isometry. $\square$

Proposition 4.3. *Let M_1, M_2, M_3 be complete metric spaces and let $\iota_1 : M_1 \to M_2$, $\iota_2 : M_2 \to M_3$ be morphisms. Then*

$$\delta(\iota_2 \circ \iota_1) \leqslant \delta(\iota_1) + \delta(\iota_2).$$

Proof. Let $\iota_1 = \langle i_1, j_1 \rangle$ and $\iota_2 = \langle i_2, j_2 \rangle$. The result follows immediately by observing that

$$
\begin{aligned}
& d(i_2 \circ i_1 \circ j_1 \circ j_2(x), x) \\
& \quad \leqslant d(i_2 \circ i_1 \circ j_1 \circ j_2(x), i_2 \circ j_2(x)) + d(i_2 \circ j_2(x), x) \\
& \quad = d(i_1 \circ j_1 \circ j_2(x), j_2(x)) + d(i_2 \circ j_2(x), x) \quad [\text{because } i_2 \text{ is isometric}] \\
& \quad \leqslant \delta(\iota_1) + \delta(\iota_2). \qquad \square
\end{aligned}
$$

We can now prove the main theorem.

Theorem 4.4 (Uniqueness of fixed point). *Let $F : \mathrm{KMS}^E \to \mathrm{KMS}^E$ be a contracting functor. Then F has a unique fixed point (up to isometry), i.e. there exists a compact metric space M such that*

1. $M \cong FM$,
2. $\forall M' \in \mathrm{KMS}^E.\ FM' \cong M' \Rightarrow M \cong M'$.

Proof. Let $F : \mathrm{KMS}^E \to \mathrm{KMS}^E$ be an ε-contracting functor. By Theorem 3.2 the direct limit $(M, (\gamma_n)_n)$ of the tower $(F^n M_0, F^n \iota_0)_n$ (where M_0 is the one-point space and $\iota_0 : M_0 \to FM_0$) provides a fixed point for the functor F. Let $\kappa : M \to FM$ be the canonical isomorphism. Suppose we have another fixed point M' so that there exists an isomorphism $\lambda : M' \to FM'$.

Let $u_0 : M_0 \to M'$ be any morphism. Notice that such a morphism always exists (it is not in general unique). We define for each $n \in N$ a morphism $\tilde{\gamma}_n : F^n M_0 \to M'$ as follows:

$$\tilde{\gamma}_0 = u_0,$$

$$\tilde{\gamma}_{n+1} = \lambda^{-1} \circ F\tilde{\gamma}_n.$$

By Proposition 4.3 $\delta(\tilde{\gamma}_{n+1}) \leqslant \delta(\lambda^{-1}) + \delta(F\tilde{\gamma}_n) \leqslant \varepsilon \cdot \delta(\tilde{\gamma}_n)$. Hence

$$\lim_{n\to\infty} \delta(\tilde{\gamma}_n) = 0.$$

Let $\gamma_n = \langle \alpha_n, \beta_n \rangle$ and $\tilde{\gamma}_n = \langle \tilde{\alpha}_n, \tilde{\beta}_n \rangle$ and consider for each n the pair of functions $\langle \tilde{\alpha}_n \circ \beta_n, \alpha_n \circ \tilde{\beta}_n \rangle$, the first one from M in M' and the second one from M' in M. They are NDI since they are compositions of NDI functions. Moreover

$$d(\tilde{\alpha}_n \circ \beta_n \circ \alpha_n \circ \tilde{\beta}_n, id_{M'}) = d(\tilde{\alpha}_n \circ \tilde{\beta}_n, id_{M'}) = \delta(\tilde{\gamma}_n),$$

and

$$d(\alpha_n \circ \tilde{\beta}_n \circ \tilde{\alpha}_n \circ \beta_n, id_M) = d(\alpha_n \circ \beta_n, id_M) = \delta(\gamma_n).$$

Now, $\lim_{n\to\infty} \delta(\tilde{\gamma}_n) = 0$, as seen above, and $\lim_{n\to\infty} \delta(\gamma_n) = 0$, by the Initiality lemma 3.1. Therefore M and M' are η-isometric for all $\eta > 0$. Thus, by Lemma 4.2 and compactness of M and M', we have $M \cong M'$. □

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